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A NEW CRITERION FOR CLOSE-TO-CONVEXITY

NAOYA KOIKE (小池尚也、群馬大学)

ABSTRACT. R.Singh and S.Singh [2] obtained a new criterion for close-to-convexity. The aim of this paper is to obtain a generalized result of them.

1. Introduction.

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N = 1, 2, 3, \ldots)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$.

A function $f(z)$ in $A(p)$ is said to be p-valently starlike if and only if

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > 0 \qquad \textit{in } E.$$

A function $f(z)$ in $A(p)$ is said to be p-valently convex if and only if

$$1 + \mathrm{Re}\frac{zf''(z)}{f'(z)} > 0 \qquad \textit{in } E.$$

Further, a function $f(z)$ in $A(p)$ is said to be p-valently close-to-convex if there exists a p-valently starlike function $g(z) \in A(p)$ such that

$$\mathrm{Re}\frac{zf'(z)}{g(z)} > 0 \qquad \textit{in } E.$$

It is well known that a p-valently close-to-convex function is p-valent in E. [4.Theorem 1]

2. Preliminary.

Lemma 1 *Let $f(z) \in A(p)$ and if there exists a (p-k+1)-valently starlike function $g(z) = \sum_{n=p-k+1}^{\infty} b_n z^n, (b_{p-k+1} \neq 0)$ that satisfies*

$$\mathrm{Re}\frac{zf^{(k)}(z)}{g(z)} > 0 \qquad \textit{in } E,$$

then $f(z)$ is p-valently close-to-convex and therefore, $f(z)$ is p-valent in E.

We can obtain the above lemma from [3.Theorem 8.].

3. Main result.

Theorem 1 *Let $f(z) \in A(p)$ and suppose*

$$\mathrm{Re}\left[\frac{1+(1-2\alpha)z}{1-z}+\frac{f'(z)}{z^{p-2}}\right]<\alpha \qquad \textit{in } E$$

for some α $(1<\alpha)$, then $f(z)$ is p-valently close-to-convex in E.

Proof. Since

$$\alpha-\frac{1+(1-2\alpha)z}{1-z}-\frac{f'(z)}{z^{p-2}}=(\alpha-1)(1+c_1z+\ldots),$$

using Herglotz representation formula [1],we have

$$\alpha-\frac{1+(1-2\alpha)z}{1-z}-\frac{f'(z)}{z^{p-2}}=(\alpha-1)\int_0^{2\pi}\frac{1+ze^{it}}{1-ze^{it}}d\mu(t), \tag{1}$$

where $\mu(t)$ is the probability measure on $[0,2\pi]$,satisfying

$$\int_0^{2\pi}d\mu(t)=1.$$

From (1) we get

$$(1-z)^2\frac{f'(z)}{z^{p-1}}=2(\alpha-1)\int_0^{2\pi}\frac{(1-z)(1-e^{it})}{1-ze^{it}}d\mu(t). \tag{2}$$

Now letting

$$K(z,t)=\frac{(1-z)(1-e^{it})}{1-ze^{it}}, \qquad (z\in E;t\in[0,2\pi])$$

then we verify that

$$\begin{aligned}\mathrm{Re}K(z,t) &= \mathrm{Re}\left[\frac{(1-re^{i\theta})(1-e^{it})}{1-re^{i(\theta+t)}}\right]\\ &= \frac{(1-\cos t)(1-r^2)}{1-2r\cos(\theta+t)+r^2}\\ &\geq 0 \qquad\qquad (z=re^{i\theta})\end{aligned}$$

for all z in E and for all t in $[0,2\pi]$.
From (2) we obtain that

$$\mathrm{Re}\left[(1-z)^2\frac{f'(z)}{z^{p-1}}\right]\geq 0 \qquad \textit{in } E.$$

Letting

$$g(z)=\frac{z^p}{(1-z)^2},$$

we see that $g(z)$ is a p-valently starlike function.
Therefore, from Lemma 1 $f(z)$ is p-valently close-to-convex in E.
This completes our proof.

Corollary 1 *Let $f(z) \in A(1)$ and suppose*

$$\mathrm{Re}\left[\frac{1+(1-2\alpha)z}{1-z} + zf'(z)\right] < \alpha \qquad in\ E$$

for some α $(1 < \alpha)$, then f(z) is close-to-convex in E.

Theorem 2 *Let $f(z) \in A(p)$ and suppose*

$$\mathrm{Re}\left[\frac{1+(1-2\alpha)z}{1-z} - \frac{f'(z)}{z^{p-2}}\right] > \alpha \qquad in\ E$$

for some α $(0 \le \alpha < 1)$,then $f(z)$ is p-valently close-to-convex in E.

Proof. Applying the same method as the proof of Theorem 1, we have

$$(1-z)^2\frac{f'(z)}{z^{p-1}} = 2(1-\alpha)\int_0^{2\pi}\frac{(1-z)(1-e^{it})}{1-ze^{it}}d\mu(t).$$

From the same reason as the proof of Theorem 1, we have

$$\mathrm{Re}\left[(1-z)^2\frac{f'(z)}{z^{p-1}}\right] > 0 \qquad in\ E.$$

This shows that $f(z)$ is p-valently close-to-convex in E.

Corollary 2 ([2]) *Let $f(z) \in A(1)$ and suppose*

$$\mathrm{Re}\left[\frac{1+(1-2\alpha)z}{1-z} - zf'(z)\right] > \alpha \qquad in\ E$$

for some α $(0 \le \alpha < 1)$,then $f(z)$ is close-to-convex in E.

Theorem 3 *Let $f(z) \in A(1)$,$g(z)$ be analytic in E with $g(0) = 1, \mathrm{Re}g'(0) > -2(1-\alpha)$ and $\mathrm{Re}g(z) \le \alpha$ $(z \in E)$ for some α $(\alpha > 1)$.*
If

$$\mathrm{Re}\,(g(z) - zf'(z)) < \alpha \qquad in\ E,$$

then $f(z)$ is close-to-convex.

Proof. Applying the same method as the proof of Theorem 1,we have

$$(1-z)^2f'(z) = \int_0^{2\pi}(1-z)^2\left[\frac{g(z)-\alpha}{z} - \frac{(1-\alpha)(1+ze^{it})}{z(1-ze^{it})}\right]d\mu(t).$$

Let

$$K(z,t) = (1-z)^2\left[\frac{g(z)-\alpha}{z} - \frac{(1-\alpha)(1+ze^{it})}{z(1-ze^{it})}\right],$$

where $z \in E$,$t \in [0, 2\pi]$,$\alpha > 1$.

Since

$$K(z,t) = (1-z)^2\left[\frac{g(z)-1}{z} - \frac{2(1-\alpha)e^{it}}{1-ze^{it}}\right],$$

$$\begin{aligned}\mathrm{Re}K(0,t) &= \mathrm{Re}\left[g'(0) - 2(1-\alpha)e^{it}\right] \\ &\geq \mathrm{Re}g'(0) + 2(1-\alpha) \\ &> 0\end{aligned}$$

Further,since

$$K(z,t) = \frac{(1-z)^2}{z}\left[g(z) - 1 - \frac{2(1-\alpha)ze^{it}}{1-ze^{it}}\right],$$

$$\begin{aligned}\mathrm{Re}K(e^{i\theta},t) &= 2(\cos\theta - 1)\mathrm{Re}\left[g(e^{i\theta}) - 1 - \frac{2(1-\alpha)e^{i(t+\theta)}}{1-e^{i(t+\theta)}}\right] \\ &= 2(\cos\theta-1)\left[\mathrm{Re}g(e^{i\theta}) - 1 + (1-\alpha)\right] \\ &= 2(\cos\theta-1)\left[\mathrm{Re}g(e^{i\theta}) - \alpha\right] \\ &\geq 0\end{aligned}$$

Therefore,

$$\mathrm{Re}\left[(1-z)^2 f'(z)\right] > 0 \qquad in\ E,$$

so $f(z)$ is close-to-convex in E.

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Department of Mathematics, University of Gunma, Aramaki, Maebashi 371, Japan.